3.4.1 EXERCISES

For a link to all of the additional resources available for this section, click OSttS Chapter 3 materials.

In Exercises 1 - 10, use the given complex numbers z and w to find and simplify the following. Write your answers in the form $a + bi$.

- $z + w$
- zw
- z^2
- $\frac{1}{z}$
- $\frac{z}{w}$
- $\frac{w}{z}$
- $\overline{z}$
- $z\overline{z}$
- $(\overline{z})^2$

For help with these exercises, click one or more of the resources below:

- Review of Complex Numbers
- Complex Conjugates

1. $z = 2 + 3i$, $w = 4i$
2. $z = 1 + i$, $w = -i$
3. $z = i$, $w = -1 + 2i$
4. $z = 4i$, $w = 2 - 2i$
5. $z = 3 - 5i$, $w = 2 + 7i$
6. $z = -5 + i$, $w = 4 + 2i$
7. $z = \sqrt{2} - i\sqrt{2}$, $w = \sqrt{2} + i\sqrt{2}$
8. $z = 1 - i\sqrt{3}$, $w = -1 - i\sqrt{3}$
9. $z = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$, $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
10. $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$, $w = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$

In Exercises 11 - 18, simplify the quantity.

For help with these exercises, click on the resource below:

- Review of Complex Numbers

11. $\sqrt{-49}$
12. $\sqrt{-9}$
13. $\sqrt{-25}\sqrt{-4}$
14. $\sqrt{(-25)(-4)}$
15. $\sqrt{-9}\sqrt{-16}$
16. $\sqrt{(-9)(-16)}$
17. $\sqrt{-(-9)}$
18. $-\sqrt{(-9)}$

We know that $i^2 = -1$ which means $i^3 = i^2 \cdot i = (-1) \cdot i = -i$ and $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$. In Exercises 19 - 26, use this information to simplify the given power of i.

For help with these exercises, click on the resource below:

- The powers of i

19. i^5
20. i^6
21. i^7
22. i^8
23. i^{15}
24. i^{26}
25. i^{117}
26. i^{304}

In Exercises 27 - 48, find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.

For help with these exercises, click one or more of the resources below:

- Finding the complex zeros of a polynomial function
- Using the Conjugate Pairs Theorem

27. $f(x) = x^2 - 4x + 13$
28. $f(x) = x^2 - 2x + 5$
29. $f(x) = 3x^2 + 2x + 10$
30. $f(x) = x^3 - 2x^2 + 9x - 18$
31. $f(x) = x^3 + 6x^2 + 6x + 5$
32. $f(x) = 3x^3 - 13x^2 + 43x - 13$
33. $f(x) = x^3 + 3x^2 + 4x + 12$
34. $f(x) = 4x^3 - 6x^2 - 8x + 15$
35. $f(x) = x^3 + 7x^2 + 9x - 2$
36. $f(x) = 9x^3 + 2x + 1$
37. $f(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3$
38. $f(x) = 2x^4 - 7x^3 + 14x^2 - 15x + 6$
39. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18$
40. $f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12$
41. $f(x) = -3x^4 - 8x^3 - 12x^2 - 12x - 5$
42. $f(x) = 8x^4 + 50x^3 + 43x^2 + 2x - 4$
43. $f(x) = x^4 + 9x^2 + 20$
44. $f(x) = x^4 + 5x^2 - 24$
45. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$
46. $f(x) = x^6 - 64$
47. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26$ (Hint: $x = i$ is one of the zeros.)
48. $f(x) = 2x^4 + 5x^3 + 13x^2 + 7x + 5$ (Hint: $x = -1 + 2i$ is a zero.)

In Exercises 49 - 53, create a polynomial f with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.

For help with these exercises, click on the resource below:

- Forming a polynomial given information about its zeros

49.
- The zeros of f are $c = \pm 1$ and $c = \pm i$
- The leading term of $f(x)$ is $42x^4$

50.
- $c = 2i$ is a zero.
- the point $(-1, 0)$ is a local minimum on the graph of $y = f(x)$
- the leading term of $f(x)$ is $117x^4$

51.
- The solutions to $f(x) = 0$ are $x = \pm 2$ and $x = \pm 7i$
- The leading term of $f(x)$ is $-3x^5$
- The point $(2, 0)$ is a local maximum on the graph of $y = f(x)$.

52.
- f is degree 5.
- $x = 6$, $x = i$ and $x = 1 - 3i$ are zeros of f
- as $x \to -\infty$, $f(x) \to \infty$

53.
- The leading term of $f(x)$ is $-2x^3$
- $c = 2i$ is a zero
- $f(0) = -16$

54. Let z and w be arbitrary complex numbers. Show that $\overline{z}\,\overline{w} = \overline{zw}$ and $\overline{\overline{z}} = z$.

Checkpoint Quiz 3.4

Let $p(x) = 2x^4 - 4x^3 + 7x^2 + 6x - 15$.

1. Given that $x = 1 - 2i$ is a zero, find the remaining zeros of p.
2. Factor $p(x)$ over the complex numbers.
3. Factor $p(x)$ over the real numbers.

For worked out solutions to this quiz, click the link below:

- Quiz Solution

3.4.2 Answers

1. For $z = 2 + 3i$ and $w = 4i$
   - $z + w = 2 + 7i$
   - $zw = -12 + 8i$
   - $z^2 = -5 + 12i$
   - $\frac{1}{z} = \frac{2}{13} - \frac{3}{13}\,i$
   - $\frac{z}{w} = \frac{3}{4} - \frac{1}{2}\,i$
   - $\frac{w}{z} = \frac{12}{13} + \frac{8}{13}\,i$
   - $\overline{z} = 2 - 3i$
   - $z\overline{z} = 13$
   - $(\overline{z})^2 = -5 - 12i$
2. For $z = 1 + i$ and $w = -i$
   - $z + w = 1$
   - $zw = 1 - i$
   - $z^2 = 2i$
   - $\frac{1}{z} = \frac{1}{2} - \frac{1}{2}\,i$
   - $\frac{z}{w} = -1 + i$
   - $\frac{w}{z} = -\frac{1}{2} - \frac{1}{2}\,i$
   - $\overline{z} = 1 - i$
   - $z\overline{z} = 2$
   - $(\overline{z})^2 = -2i$
3. For $z = i$ and $w = -1 + 2i$
   - $z + w = -1 + 3i$
   - $zw = -2 - i$
   - $z^2 = -1$
   - $\frac{1}{z} = -i$
   - $\frac{z}{w} = \frac{2}{5} - \frac{1}{5}\,i$
   - $\frac{w}{z} = 2 + i$
   - $\overline{z} = -i$
   - $z\overline{z} = 1$
   - $(\overline{z})^2 = -1$
4. For $z = 4i$ and $w = 2 - 2i$
   - $z + w = 2 + 2i$
   - $zw = 8 + 8i$
   - $z^2 = -16$
   - $\frac{1}{z} = -\frac{1}{4}\,i$
   - $\frac{z}{w} = -1 + i$
   - $\frac{w}{z} = -\frac{1}{2} - \frac{1}{2}\,i$
   - $\overline{z} = -4i$
   - $z\overline{z} = 16$
   - $(\overline{z})^2 = -16$
5. For $z = 3 - 5i$ and $w = 2 + 7i$
   - $z + w = 5 + 2i$
   - $zw = 41 + 11i$
   - $z^2 = -16 - 30i$
   - $\frac{1}{z} = \frac{3}{34} + \frac{5}{34}\,i$
   - $\frac{z}{w} = -\frac{29}{53} - \frac{31}{53}\,i$
   - $\frac{w}{z} = -\frac{29}{34} + \frac{31}{34}\,i$
   - $\overline{z} = 3 + 5i$
   - $z\overline{z} = 34$
   - $(\overline{z})^2 = -16 + 30i$

6. For $z = -5 + i$ and $w = 4 + 2i$

   - $z + w = -1 + 3i$
   - $zw = -22 - 6i$
   - $z^2 = 24 - 10i$
   - $\frac{1}{z} = -\frac{5}{26} - \frac{1}{26}\,i$
   - $\frac{z}{w} = -\frac{9}{10} + \frac{7}{10}\,i$
   - $\frac{w}{z} = -\frac{9}{13} - \frac{7}{13}\,i$
   - $\overline{z} = -5 - i$
   - $z\overline{z} = 26$
   - $(\overline{z})^2 = 24 + 10i$

7. For $z = \sqrt{2} - i\sqrt{2}$ and $w = \sqrt{2} + i\sqrt{2}$

   - $z + w = 2\sqrt{2}$
   - $zw = 4$
   - $z^2 = -4i$
   - $\frac{1}{z} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}\,i$
   - $\frac{z}{w} = -i$
   - $\frac{w}{z} = i$
   - $\overline{z} = \sqrt{2} + i\sqrt{2}$
   - $z\overline{z} = 4$
   - $(\overline{z})^2 = 4i$

8. For $z = 1 - i\sqrt{3}$ and $w = -1 - i\sqrt{3}$

   - $z + w = -2i\sqrt{3}$
   - $zw = -4$
   - $z^2 = -2 - 2i\sqrt{3}$
   - $\frac{1}{z} = \frac{1}{4} + \frac{\sqrt{3}}{4}\,i$
   - $\frac{z}{w} = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
   - $\frac{w}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
   - $\overline{z} = 1 + i\sqrt{3}$
   - $z\overline{z} = 4$
   - $(\overline{z})^2 = -2 + 2i\sqrt{3}$

9. For $z = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$ and $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$

   - $z + w = i\sqrt{3}$
   - $zw = -1$
   - $z^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
   - $\frac{1}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
   - $\frac{z}{w} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
   - $\frac{w}{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
   - $\overline{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
   - $z\overline{z} = 1$
   - $(\overline{z})^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$

10. For $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$ and $w = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$

    - $-\sqrt{2}$
    - $zw = 1$
    - $z^2 = -i$
    - $\frac{1}{z} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$
    - $\frac{z}{w} = -i$
    - $\frac{w}{z} = i$
    - $\overline{z} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i$
    - $z\overline{z} = 1$
    - $(\overline{z})^2 = i$

11. $7i$ 12. $3i$ 13. -10 14. 10

15. -12 16. 12 17. 3 18. $-3i$

19. $i^5 = i^4 \cdot i = 1 \cdot i = i$

20. $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

21. $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

22. $i^8 = i^4 \cdot i^4 = \left(i^4\right)^2 = (1)^2 = 1$

23. $i^{15} = \left(i^4\right)^3 \cdot i^3 = 1 \cdot (-i) = -i$

24. $i^{26} = \left(i^4\right)^6 \cdot i^2 = 1 \cdot (-1) = -1$

25. $i^{117} = \left(i^4\right)^{29} \cdot i = 1 \cdot i = i$

26. $i^{304} = \left(i^4\right)^{76} = 1^{76} = 1$

27. $f(x) = x^2 - 4x + 13 = (x - (2 + 3i))(x - (2 - 3i))$
Zeros: $x = 2 \pm 3i$

28. $f(x) = x^2 - 2x + 5 = (x - (1 + 2i))(x - (1 - 2i))$
Zeros: $x = 1 \pm 2i$

29. $f(x) = 3x^2 + 2x + 10 = 3\left(x - \left(-\frac{1}{3} + \frac{\sqrt{29}}{3}i\right)\right)\left(x - \left(-\frac{1}{3} - \frac{\sqrt{29}}{3}i\right)\right)$
Zeros: $x = -\frac{1}{3} \pm \frac{\sqrt{29}}{3}i$

30. $f(x) = x^3 - 2x^2 + 9x - 18 = (x - 2)\left(x^2 + 9\right) = (x - 2)(x - 3i)(x + 3i)$
Zeros: $x = 2, \pm 3i$

31. $f(x) = x^3 + 6x^2 + 6x + 5 = (x+5)(x^2+x+1) = (x+5)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$
Zeros: $x = -5,\ x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

32. $f(x) = 3x^3 - 13x^2 + 43x - 13 = (3x - 1)(x^2 - 4x + 13) = (3x - 1)(x - (2 + 3i))(x - (2 - 3i))$
Zeros: $x = \frac{1}{3},\ x = 2 \pm 3i$

33. $f(x) = x^3 + 3x^2 + 4x + 12 = (x + 3)\left(x^2 + 4\right) = (x + 3)(x + 2i)(x - 2i)$
Zeros: $x = -3,\ \pm 2i$

34. $f(x) = 4x^3 - 6x^2 - 8x + 15 = \left(x + \frac{3}{2}\right)\left(4x^2 - 12x + 10\right)$
$= 4\left(x + \frac{3}{2}\right)\left(x - \left(\frac{3}{2} + \frac{1}{2}i\right)\right)\left(x - \left(\frac{3}{2} - \frac{1}{2}i\right)\right)$
Zeros: $x = -\frac{3}{2},\ x = \frac{3}{2} \pm \frac{1}{2}i$

35. $f(x) = x^3 + 7x^2 + 9x - 2 = (x + 2)\left(x - \left(-\frac{5}{2} + \frac{\sqrt{29}}{2}\right)\right)\left(x - \left(-\frac{5}{2} - \frac{\sqrt{29}}{2}\right)\right)$
Zeros: $x = -2,\ x = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}$

36. $f(x) = 9x^3 + 2x + 1 = \left(x + \frac{1}{3}\right)\left(9x^2 - 3x + 3\right)$
$= 9\left(x + \frac{1}{3}\right)\left(x - \left(\frac{1}{6} + \frac{\sqrt{11}}{6}i\right)\right)\left(x - \left(\frac{1}{6} - \frac{\sqrt{11}}{6}i\right)\right)$
Zeros: $x = -\frac{1}{3},\ x = \frac{1}{6} \pm \frac{\sqrt{11}}{6}i$

37. $f(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3 = \left(x - \frac{1}{2}\right)^2\left(4x^2 + 12\right) = 4\left(x - \frac{1}{2}\right)^2(x + i\sqrt{3})(x - i\sqrt{3})$
Zeros: $x = \frac{1}{2},\ x = \pm\sqrt{3}i$

38. $f(x) = 2x^4 - 7x^3 + 14x^2 - 15x + 6 = (x-1)^2\left(2x^2 - 3x + 6\right)$
$= 2(x-1)^2\left(x - \left(\frac{3}{4} + \frac{\sqrt{39}}{4}i\right)\right)\left(x - \left(\frac{3}{4} - \frac{\sqrt{39}}{4}i\right)\right)$
Zeros: $x = 1,\ x = \frac{3}{4} \pm \frac{\sqrt{39}}{4}i$

39. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18 = (x+2)(x-1)\left(x^2+9\right) = (x+2)(x-1)(x+3i)(x-3i)$
Zeros: $x = -2,\ 1,\ \pm 3i$

40. $f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12 = 6\left(x + \frac{1}{3}\right)\left(x - \frac{3}{2}\right)\left(x - \left(-2 + 2\sqrt{2}\right)\right)\left(x - \left(-2 - 2\sqrt{2}\right)\right)$
Zeros: $x = -\frac{1}{3},\ x = \frac{3}{2},\ x = -2 \pm 2\sqrt{2}$

41. $f(x) = -3x^4 - 8x^3 - 12x^2 - 12x - 5 = (x+1)^2\left(-3x^2 - 2x - 5\right)$
$= -3(x+1)^2\left(x - \left(-\frac{1}{3} + \frac{\sqrt{14}}{3}i\right)\right)\left(x - \left(-\frac{1}{3} - \frac{\sqrt{14}}{3}i\right)\right)$
Zeros: $x = -1,\ x = -\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

42. $f(x) = 8x^4 + 50x^3 + 43x^2 + 2x - 4 = 8\left(x + \frac{1}{2}\right)\left(x - \frac{1}{4}\right)(x - (-3 + \sqrt{5}))(x - (-3 - \sqrt{5}))$
Zeros: $x = -\frac{1}{2},\ \frac{1}{4},\ x = -3 \pm \sqrt{5}$

43. $f(x) = x^4 + 9x^2 + 20 = \left(x^2+4\right)\left(x^2+5\right) = (x-2i)(x+2i)\left(x - i\sqrt{5}\right)\left(x + i\sqrt{5}\right)$
Zeros: $x = \pm 2i, \pm i\sqrt{5}$

44. $f(x) = x^4 + 5x^2 - 24 = \left(x^2-3\right)\left(x^2+8\right) = (x - \sqrt{3})(x + \sqrt{3})\left(x - 2i\sqrt{2}\right)\left(x + 2i\sqrt{2}\right)$
Zeros: $x = \pm\sqrt{3}, \pm 2i\sqrt{2}$

45. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12 = (x-1)\left(x^2+3\right)\left(x^2+4\right)$
$= (x-1)(x - i\sqrt{3})(x + i\sqrt{3})(x-2i)(x+2i)$
Zeros: $x = 1,\ \pm\sqrt{3}i,\ \pm 2i$

46. $f(x) = x^6 - 64 = (x-2)(x+2)\left(x^2 + 2x + 4\right)\left(x^2 - 2x + 4\right)$
$= (x-2)(x+2)\left(x - \left(-1 + i\sqrt{3}\right)\right)\left(x - \left(-1 - i\sqrt{3}\right)\right)\left(x - \left(1 + i\sqrt{3}\right)\right)\left(x - \left(1 - i\sqrt{3}\right)\right)$
Zeros: $x = \pm 2,\ x = -1 \pm i\sqrt{3},\ x = 1 \pm i\sqrt{3}$

47. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26 = (x^2 - 2x + 26)(x^2+1) = (x - (1+5i))(x - (1-5i))(x+i)(x-i)$
Zeros: $x = 1 \pm 5i,\ x = \pm i$

48. $f(x) = 2x^4 + 5x^3 + 13x^2 + 7x + 5 = \left(x^2 + 2x + 5\right)\left(2x^2 + x + 1\right)$
$= 2(x - (-1 + 2i))(x - (-1 - 2i))\left(x - \left(-\frac{1}{4} + i\frac{\sqrt{7}}{4}\right)\right)\left(x - \left(-\frac{1}{4} - i\frac{\sqrt{7}}{4}\right)\right)$
Zeros: $x = -1 \pm 2i, -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$

49. $f(x) = 42(x-1)(x+1)(x-i)(x+i)$

50. $f(x) = 117(x+1)^2(x-2i)(x+2i)$

51. $f(x) = -3(x-2)^2(x+2)(x-7i)(x+7i)$

52. $f(x) = a(x-6)(x-i)(x+i)(x-(1-3i))(x-(1+3i))$ where a is any real number, $a < 0$

53. $f(x) = -2(x-2i)(x+2i)(x+2)$